**TUNGUSHBAYEVA INDIRA ORAZBEKOVNA**

# «MODEL-THEORETIC PROPERTIES OF ALGEBRAS WHOSE THEORIES ARE JONSSON»

## ABSTRACT

**of the dissertation submitted for the degree of Doctor of Philosophy (PhD) in the educational program 8D05401 - "Mathematics".**

**Relevance of the topic.** This dissertation is presented as the result of scientific research related to one of the modern directions in fundamental mathematics – Model Theory. Model Theory is a branch of mathematics that emerged from the combination of methods from Abstract Algebra, Universal Algebra, Mathematical Logic, Algebraic Geometry, and Topology. The primary subjects of Model Theory are formal languages and their models. The main objective of Model Theory is to describe the fundamental relationships between the syntax and semantics of the formal language under consideration (in our case, this is the first-order predicate calculus).

Traditionally, Model Theory has been divided into two directions, commonly referred to as the "Eastern" and "Western" approaches. These names are linked to the geographical locations of the founders of Model Theory, namely Alfred Tarski and Abraham Robinson. As is well known, A. Tarski resided on the west coast of the United States, while A. Robinson lived on the east coast. Research based on the methods of the "Western" Model Theory is typically associated with issues in mathematical analysis and algebraic geometry, and assumes completeness of the language used. In contrast, the "Eastern" Model Theory evolved from the current problems in Universal Algebra and various subfields of Abstract Algebra. This dissertation belongs to the "Eastern" approach in Model Theory and focuses on the study of incomplete theories, specifically the class of Jonsson theories.

Jonsson theories constitute a special subclass of inductive theories, characterized by the following model-theoretic properties: the existence of infinite models, the amalgamation property, and the joint embedding property. It is important to note that, by definition, Jonsson theories are not generally complete. Given that the class of incomplete theories is much broader than the class of complete theories, it becomes clear that the study of incomplete theories and their models is a more complex and highly relevant task. However, due to the absence of the property of elementary equivalence between models of incomplete theories and other model-theoretic properties, such a task seems insurmountable. This is why the focus of the research was on the class of inductive theories, within which we concentrated on studying Jonsson theories. The class of Jonsson theories includes such theories as group theory, theory of abelian groups, theory of fields of fixed characteristic, theory of linear orders, and many others. These theories are incomplete; however, there are also examples of complete Jonsson theories, such as the theory of divisible abelian groups, the theory of algebraically closed fields of fixed characteristic, and the theory of dense linear orders without endpoints, among others. Thus, the chosen class is not only represented by a sufficient number of classical examples but is also quite diverse, making it interesting for further model-theoretic investigations.

In this work, new tools are presented for studying Jonsson theories, both of a semantic and syntactic nature. The dissertation research was conducted along two directions.

The first direction originates from the study of classical algebraic structures in Differential Algebra from the perspective of investigating Jonsson theories. New examples of Jonsson theories are presented, including the theory $DF_0$ of differential fields of characteristic 0, the theory $DCF_0$ of differentially closed fields of characteristic 0, the theory $DPF_p$ of differentially perfect fields of characteristic $p$, and the theory $DCF_p$ of differentially closed fields of characteristic $p$. It is noted that the theory $DF_p$ of differential fields of characteristic $p$ is not Jonsson. The reason for this is the absence of the amalgamation property in this theory, which is of interest in the context of studying the properties of amalgamation and joint embedding and their interrelationship. During further investigation, we focused on the connection between these properties in the model classes of the theories $DF_0$ and $DPF_p$. To generalize the results obtained in studying the mentioned properties of these theories, special subclasses of inductive theories were defined in relation to the model-theoretic connection between the properties of amalgamation and joint embedding. Various examples were constructed, and some sufficient conditions describing this connection were provided.

The second direction of the work relates to the expansion of the tools for studying Jonsson theories within the context of applying the so-called semantic method, which first appeared in the works of T.G. Mustafin. This method involves the study of the properties of Jonsson theories through the characterization of their semantic models, and usually implies the construction of the Jonsson spectrum for a given class of structures in a countable first-order language. The further application of this method to study the properties of many specific Jonsson theories and classes of their models was developed in the works of A.R. Yeshkeyev. To refine and expand the tools for studying Jonsson theories, A.R. Yeshkeyev introduced several new definitions, including the definition of the Jonsson spectrum. The concept of the Jonsson spectrum is the primary tool in the semantic method. Therefore, obtaining various results that describe the Jonsson spectrum of a fixed class of structures is of great significance for the further development and refinement of the tools for studying Jonsson theories and their classes of models. This dissertation demonstrates the application of the semantic method in studying the properties of the cosemanticness classes of Jonsson theories, the Jonsson spectrum for a fixed class of structures of the considered language, and presents results on the construction of certain types of algebraic structures on the Jonsson spectrum and the cosemanticness classes of this spectrum.

Based on the above, it can be concluded that the study of Jonsson theories remains relevant in light of the recent advancements in Model Theory, which in turn leads to the development of new methods for investigating both complete and incomplete theories, as well as the refinement of existing mathematical research tools.

**The purpose of the work.** The aim of this dissertation research is to study the model-theoretic properties of fixed theories in relation to a priori defined logical relationships between the fundamental concepts that define the Jonsson theories discussed in the dissertation. These include the following properties: joint embedding, amalgamation, axiomatization, completeness, categoricity, perfectness, convexity, algebraization of the cosemanticness classes of the considered Jonsson spectrum, and various interrelations between these classes.

**Research objectives.** Within the framework of this dissertation research, the following tasks were set:

1. To find new examples of Jonsson theories among classical structures in differential algebra and demonstrate their perfectness;
2. To obtain a sufficient condition for Jonssonness for complete theories in relation to the property of categoricity and the properties of the class of existentially closed models;
3. To obtain sufficient conditions for a given theory $T$ to be an AP-theory;
4. To describe the structure of the Robinson spectrum for any class of $L$-structures;
5. To find a sufficient condition for the finiteness of the cosemanticness class of a fixed Jonsson theory;
6. To show the connection between the class of existentially axiomatizable theories and the class of Jonsson theories using the semantic method;
7. To describe the algebraic structure of the Jonsson spectrum and the cosemanticness classes of the Jonsson spectrum for a fixed class of $L$-structures.

**The objects of research.** The primary object of this dissertation research is the fixed Jonsson theories and their model classes.

**Subject of research.** This dissertation investigates the model-theoretic properties of Jonsson theories and their model classes, including the formulas of these theories in relation to their models.

**Research methodology.** The main methods used in this dissertation research are both classical methods from Mathematical Logic and Universal Algebra, as well as modern techniques specifically applied in the study of Jonsson theories.

Classical methods of Mathematical Logic refer to the use of the technical syntactical apparatus of first-order predicate logic. Methods of Universal Algebra primarily involve the semantic approach to studying various algebraic structures, such as groups, abelian groups, rings, fields, and lattices.

As for the methodology applied to the study of Jonsson theories, we distinguish between the method of transferring properties from complete theories to the case of incomplete theories, as well as the semantic method. Both of these methods were proposed by A.R. Yeshkeyev.

Since Jonsson theories are, in general, incomplete, the tools for studying them are somewhat limited. Therefore, selecting appropriate methods, typically used for studying complete theories, to derive results about Jonsson theories is not only reasonable but also fully justified by the fact that the center of Jonsson theory is a complete theory, which allows the transfer of some important model-theoretic properties from the core to the theory itself.

The essence of the semantic method is to study the properties of the semantic model as a semantic invariant of a given Jonsson theory. This method makes it possible to characterize not only the theory itself but also an entire class of theories that are cosemantic to it, both semantically and syntactically. As mentioned earlier, the key tool of the semantic method is the concept of the Jonsson spectrum of a class of structures for a fixed language. The basic algorithm for applying the semantic method generally follows these steps:

1. Selecting an appropriate class of structures for the given first-order language.
2. Constructing the Jonsson spectrum for this class of structures, with the Robinson spectrum being a special case.
3. Introducing the relation of co-semantics for Jonsson theories on the constructed Jonsson spectrum.
4. Choosing an arbitrary co-semantics class or a co-semantics class that satisfies certain specified conditions.
5. Studying the properties of the theories in this class through the properties of the semantic model of the given co-semantics class.

This dissertation not only demonstrates the application of the semantic method but also provides results that enhance its arsenal and expand its potential for deriving new model-theoretic results.

**Main results.** The following key results are presented for defense:

1. Theories $DF_0$, $DCF_0$, $DPF_p$, and $DCF_p$ are perfect Jonsson theories.
2. Let $T$ be a complete $\omega$-categorical theory, whose class of existentially closed models is non-empty. Then $T$ is a complete Jonsson theory.
3. Let an $L$-theory $T$ be strongly convex, and let $T$ admit the amalgamation property. Then $T$ is an AP-theory.
4. Let $T$ be a Robinsonian theory, and $T'$ be a Jonsson theory that is cosemantic with $T$. Then $T \subseteq T'$.
5. Let $K$ be any class of $L$-structures (possibly containing only one structure), and $RSp(K)_{/\bowtie}$ be the factor-set of the Robinson spectrum of $K$ with respect to cosemanticness relations. Then each cosemanticness class $[\varDelta]$ contains exactly one theory. In other words, for any two Robinsonian theories $T$ and $T'$ of the language $L$, the cosemanticness relation is equivalent to logical equivalence of theories, i.e. $T \bowtie T' \Leftrightarrow T = T'$.
6. Let $T$ be a Jonsson theory, and $T^0 = Th_{\forall\exists}(C_{[T]})$ be a finitely axiomatizable theory. Then there are finitely many theories cosemantic with $T$.
7. Any existentially axiomatizable theory is a Jonsson theory which is cosemantic with the empty theory.
8. Let $T_1$, $T_2$ be Jonsson theories of the language $L$, such that the set of sentences $T_1 \cup T_2$ is consistent and there exists at least one model $M \in Mod(T_1 \cup T_2)$ of infinite cardinality. Then $T_1 \cup T_2$ is a Jonsson theory.

9. Let $L$ be a first-order language, and let $K$ be a class of $L$-structures containing at least one infinite $L$-structure, with $JSp(K)$ being the Jonsson spectrum of $K$. Then $(JSp(K),\cup)$ is a commutative monoid.

10. For any co-semantical class $[T] \in JSp(K)_{/\bowtie}$, where $K$ is a class of $L$-structures containing at least one infinite structure, $[T]$ forms a lattice with respect to the operations "$\vee$" and "$\wedge$".

**Description of the main results of the study.** The volume of the presented dissertation is 82 pages. The results of the conducted research are presented in two chapters of the dissertation.

Chapter 1: "Necessary Elements from Algebra and Related Model Theory" consists of 5 sections and provides the theoretical background required to understand the content of this work. Section 1.1 presents information about classical algebraic structures relevant to this research. Section 1.2 defines concepts and discusses known results from classical Model Theory. Sections 1.3-1.5 offer a model-theoretic description of certain differential algebras relevant to the dissertation.

Chapter 2: "Jonsson Theories" consists of 6 sections and is dedicated to the study of Jonsson theories. It provides the necessary information about this class of theories and presents the results obtained during the research. Section 2.1 outlines the basic information on Jonsson theories from both classical and modern model-theoretic perspectives. Sections 2.2-2.3 present the results concerning differential fields obtained in this research, as well as describe certain properties of subclasses of Jonsson theories related to the study of amalgamation and joint embedding properties, which include the mentioned differential algebras. Sections 2.4-2.6 present the research of the research on the classes of cosemantic Jonsson theories and the algebraic structure of the Jonsson spectrum, aimed at expanding the semantic method toolkit for further study of Jonsson and inductive theories and their model classes.

**Substantiation of the novelty and importance of the results obtained.** During the course of the dissertation research, the following concepts were introduced to study the model-theoretic properties of fixed Jonsson theories: AP-theory, JEP-theory, and AJ-theory. These concepts are new and have not been proposed by other authors before, yet they have proven to be necessary and useful in expanding the toolkit for studying not only Jonsson theories but also the theories of the considered language as a whole. Additionally, all the results obtained within the framework of this dissertation research are theorems that have not been previously published in the works of other authors. They demonstrate a new, fresh approach in the development of the theory of models and related fields.

This dissertation belongs to the field of fundamental mathematics and thus has a theoretical nature. The results obtained can be used in research related to various areas of Model Theory, Mathematical Logic, Universal Algebra, and other related disciplines.

**Approbation of the work.** The results obtained during the dissertation research were presented and validated at the following scientific conferences and seminars:

1. Traditional International April Conference in Honor of the Day of Science Workers of the Republic of Kazakhstan (Almaty, April 4-9, 2022);
2. IX International Scientific Conference "Problems of Differential Equations, Analysis, and Algebra" (Aktobe, May 24-28, 2022);
3. International Scientific Conference "Current Problems of Mathematics, Mechanics, and Informatics" dedicated to the 80th anniversary of Professor T.G. Mustafin (Karaganda, September 8-9, 2022);
4. Proceedings of the International Scientific Conference "Mathematical Logic and Computer Science" (Astana, October 7-8, 2022);
5. VII Franco-Kazakh Colloquium in Model Theory. Abstracts. Claude Bernard Lyon 1 University, Camille Jordan Institute (Lyon, November 14-18, 2022);
6. International Scientific and Practical Conference "Taimanov Readings - 2022", dedicated to the 105th anniversary of Doctor of Physical and Mathematical Sciences, Academician A.D. Taimanov, and the 90th anniversary of M.Utemisov West Kazakhstan University (Uralsk, November 30, 2022);
7. Traditional International April Conference in Honor of the Day of Science Workers of the Republic of Kazakhstan (Almaty, April 5-7, 2023);
8. LOGIC COLLOQUIUM 2023. European Summer Meeting of the Association for Symbolic Logic. University of Milan (Italy, June 5-9, 2023);
9. XIII International Conference of the Georgian Mathematical Union. Batumi Shota Rustaveli State University (Georgia, September 4-9, 2023);
10. VII World Congress of the Turkic World (TWMS Congress-2023) (Turkistan, September 20-23, 2023);
11. Traditional International April Mathematical Conference in Honor of the Day of Science Workers of the Republic of Kazakhstan (April 16-19, 2024, IMM, KazNPU named after Abai Kunanbayev, Almaty, International Mathematical Center of the Siberian Branch of the Russian Academy of Sciences, Novosibirsk)
12. International Scientific Conference of Students and Young Researchers "Gylym jane Bilim" (April 14, 2024, L.N. Gumilyov Eurasian National University, Astana)
13. International Scientific Conference "Mathematics in the Constellation of Sciences", dedicated to the 85th anniversary of Academician V.A. Sidorovich of the Russian Academy of Sciences (April 1-2, 2024, M.V. Lomonosov Moscow State University, Astana)
14. International Scientific Conference of Students, Postgraduates, and Young Researchers "Lomonosov-2024" (April 19-20, 2024, M.V. Lomonosov Moscow State University, Astana)
15. International Conference "Algebra and Mathematical Logic: Theory and Applications" (June 26 - July 2, 2024, Kazan)
16. XVI International Summer School-Conference "Frontier Issues in Model Theory and Universal Algebra" (July 8-13, 2024, IM SB RAS, Novosibirsk)
17. Joint Seminar on "Model Theory" (May 25, 2022, and May 3, 2023, RAS SB Institute of Mathematics, Novosibirsk, IMM, Almaty)
18. Traditional International Conference "Maltsev Readings" (November 11-15, 2024, IM SB RAS, Novosibirsk)

The results of the dissertation research were also regularly presented in reports at the "Model Theory" seminar of the Department of Algebra, Mathematical Logic, and Geometry, named after Professor T.G. Mustafin, at the Karaganda Buketov University.

**Compliance with the directions of scientific development or government programs.** The dissertation work was carried out within the framework of the following two projects funded by the state budget: "The cosemantic classes and their classes of models" IRN AP09260237 (implementation period: 2021–2023), and "Fragments of definable subsets of the semantic model of fixed Jonsson theory," IRN AP23489523 (implementation period: 2024–2026). The topic of the dissertation research corresponds to the priority direction «Intellectual potential of the country» in the field of science «Natural Sciences», the specialized scientific direction «Fundamental and applied research in Mathematics, Mechanics, Astronomy, Physics, Chemistry, Biology, Computer Science and Geography».

**Publications.** All results of the dissertation work have been published in 25 scientific papers, including:

– 5 articles published in journals indexed in Scopus, with a percentile of no less than 35.

– 20 abstracts of reports published in the proceedings of international and foreign conferences.

**Description of the doctoral student's contribution to the preparation of each publication.** The main results of the dissertation have been published in 5 works. The doctoral candidate's contribution lies in analyzing the sources used on the research topic, formulating the main results, and providing their proof.

**Structure and Volume of the Dissertation.** The dissertation consists of an introduction, two chapters, a conclusion, and a list of references.

**Number of References Used:** 88.

**Keywords:** Jonsson theory, semantic model, amalgamation property, joint embedding property, differential field, differentially closed field, differentially complete field, AP-theory, Jonsson spectrum, cosemantic Jonsson theories, Kaiser hull.